arXiv:math/0409193v1 [math.CV] 11 Sep 2004

On the Cycle Spaces Associated to Orbits of Semi-simple Lie Groups

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February 1, 2008

Abstract

Let G be a semi-simple Lie group and Q a parabolic subgroup of its complexification $G^{\mathbb{C}}$, then $Z = G^{\mathbb{C}}/Q$ is a compact complex $G^{\mathbb{C}}$-homogeneous manifold. The group G as well as $K^{\mathbb{C}}$, the complexification of the maximal compact subgroup of G, acts naturally on Z with finitely many orbits. For any G-orbit γ, there exists a $K^{\mathbb{C}}$-orbit κ such that the intersection $\gamma \cap \kappa$ is non-empty and compact. Considering cycle intersection at the boundary of a G-orbit, a definition of the cycle space associated to any G-orbit is given. Using methods involving Schubert varieties and Schubert slices together with geometric properties of a certain complementary incidence hypersurface, the cycle space associated to an arbitrary G-orbit γ is completely characterised. In particular, it is shown that all the cycle spaces except in a few Hermitian cases are equivalent to the domain Ω_{AG}. In the exceptional Hermitian cases, the cycle spaces are equivalent to the associated bounded domain.

1 Introduction

If G is a semi-simple Lie group and Q a parabolic subgroup of its complexification $G^{\mathbb{C}}$, then the $G^{\mathbb{C}}$-homogeneous, complex manifold $Z = G^{\mathbb{C}}/Q$ is a

*Dissertation submitted in partial fulfillment for the award of a Ph.D. degree in Mathematics at the Ruhr Universität Bochum, Germany.

projective-algebraic manifold. Moreover, G has only finitely many orbits, thus at least an open orbit in Z ([W1]. Theorem 2.6).

The complexification $K^{\mathbb{C}}$ of the maximal compact subgroup $K \subset G$ also has finitely many orbits in Z. Let $Orb_Z(G)$ (resp. $Orb_Z(K^{\mathbb{C}})$) denote the set of G-orbits (resp. $K^{\mathbb{C}}$-orbits) in Z. If $\kappa \in Orb_Z(K^{\mathbb{C}})$ and $\gamma \in Orb_Z(G)$, then (γ, κ) is said to be a dual pair if the intersection $\kappa \cap \gamma$ is non-empty and compact. Duality between G and $K^{\mathbb{C}}$-orbits is proved in ([M], see also [BL] and [MUV]):
For every $\gamma \in Orb_Z(G)$ there exists a unique $\kappa \in Orb_Z(K^{\mathbb{C}})$ such that (γ, κ) is a dual pair and vice versa.

Every open G-orbit contains a unique K-orbit which is a complex manifold (see, [W1], Lemma 5.1). This is duality for open orbits. Let D be an open G-orbit in Z, define the cycle space associated to D to be the connected component $\Omega_W(D)$ of the set

$$\{g(C_0) : g \in G^{\mathbb{C}} \text{ and } g(C_0) \subset D\},$$

where C_0 denotes the base cycle. This set is a G-invariant domain in the $G^{\mathbb{C}}$-homogeneous, complex manifold $\Omega = G^{\mathbb{C}}.C_0$ contained in the cycle space $\mathcal{C}_q(Z)$ of q-dimensional cycles in Z, where $q := dim C_0$.

By the procedure of identifying a cycle $g(C_0) \in \Omega_W(D)$ with $g \in G^{\mathbb{C}}$, one can consider that $\Omega_W(D)$ is parametrized by the group $G^{\mathbb{C}}$. In this regard, $\Omega_W(D)$ is then an open subset of $G^{\mathbb{C}}$ which is invariant by the right $K^{\mathbb{C}}$-action on $G^{\mathbb{C}}$.

Several authors have been concerned with the problem of describing cycle domains $\Omega_W(D)$ of open G-orbits in Z, (see for example, [W1], [W2], [WZ], [HW1], [HW2] and [H] among others). The cycle space $\Omega_W(D)$ associated to an open G-orbit D is well understood. With the exception of a few hermitian cases, it has been shown ([HW1],[FH]) that the cycle space $\Omega_W(D)$ for any open G-orbit D in any flag manifold Z agrees with a certain domain Ω_{AG} introduced in ([AG]) independent of D and Z. This domain Ω_{AG} is an open neighborhood of the Riemannian symmetric space G/K in $G^{\mathbb{C}}/K^{\mathbb{C}}$ on which the G-action is proper ([AG]).

It is appropriate to consider an analogous definition for the cycle space of lower-dimensional G-orbits in Z. For a dual pair $(\gamma, \kappa) \in Orb_Z(G) \times Orb_Z(K^{\mathbb{C}})$, define the *cycle space* $C(\gamma)$ associated to γ as the connected

component containing the identity of the interior of the set

$$\{g \in G^{\mathbb{C}} : g(\kappa) \cap \gamma \text{ is non-empty and compact}\}.$$

It is clear that for $\gamma = D$ an open G-orbit, $C(\gamma)$ agrees with $\Omega_W(D)$.

Apriori, however, it is not clear that $C(\gamma)$ is a non-empty open subset of $G^{\mathbb{C}}$ containing the identity. It could happen that $g(c\ell(\kappa))$ intersects the boundary $\mathrm{bd}(\gamma)$ of γ in such a way that an arbitrarily small perturbation of $g(\kappa)$ has non-compact intersection with γ. In Subsection 1.2, we prove that $C(\gamma)$ so defined above is indeed a non-empty open subset of $G^{\mathbb{C}}$ containing the identity.

Although $C(\gamma)$ is by definition an open subset of $G^{\mathbb{C}}$, one can think of elements of $C(\gamma)$ as cycles in $\mathcal{C}_q(Z)$ in the following way. Since $C(\gamma)$ is $K^{\mathbb{C}}$-invariant, one often regards it generically as being in the affine homogenous space $G^{\mathbb{C}}/K^{\mathbb{C}}$. Furthermore, the $G^{\mathbb{C}}$-action on $\mathcal{C}_q(Z)$ is algebraic and the $G^{\mathbb{C}}$-isotropy subgroup $G^{\mathbb{C}}_{C_0}$ at the base point $C_0 := c\ell(\kappa)$ contains $K^{\mathbb{C}}$. One can show that the orbit $G^{\mathbb{C}}.C_0$ is either $G^{\mathbb{C}}/\bar{K}^{\mathbb{C}}$, where $\bar{K}^{\mathbb{C}}$ is a finite extension of $K^{\mathbb{C}}$, or one of the compact Hermitian symmetric spaces $X_\pm := G^{\mathbb{C}}/P_\pm$. Here, $P_\pm$ are the unique parabolic subgroups of $G^{\mathbb{C}}$ containing $K^{\mathbb{C}}$ and $G^{\mathbb{C}}/P_-$ (resp. $G^{\mathbb{C}}/P_+$) is the compact Hermitian symmetric space dual to and containing the Hermitian bounded domain $\mathcal{B}$ (resp. $\bar{\mathcal{B}}$) as a G-orbit.

Our main aim in this work is to give a detailed description of the cycle space $C(\gamma)$ associated to any G-orbit γ. We prove the following

Theorem 1.1. *If $c\ell(\kappa)$ is either P_+- (or P_-)-invariant, then $C(\gamma)$ is the bounded symmetric domain $\mathcal{B}$ (or its complex conjugate $\bar{\mathcal{B}}$), otherwise $C(\gamma)$ agrees with the domain Ω_{AG}.*

For the proof of this result certain subvarieties are introduced and their intersections with cycles are studied. A Borel subgroup B of $G^{\mathbb{C}}$ containing the factor AN of an Iwasawa decomposition $G = KAN$ of G is referred to as an Iwasawa-Borel subgroup. The closure S of such a B-orbit $\mathcal{O}$ is called an Iwasawa-Schubert variety. A meromorphic function $f \in \Gamma(S, \mathcal{O}(*Y))$, with polar set contained in the variety $S \setminus \mathcal{O}$ is then constructed. It is shown that the polar set of its trace transform $\mathcal{P} := \mathcal{P}(Tr(f))$ is a complex B-invariant hypersurface in the complement of $C(\gamma)$. The polar set $\mathcal{P}$ consists of a union

of hypersurface components of the maximal B-invariant hypersurface H in the complement of $C(\gamma)$. This complementary B-invariant hypersurface H is decisive in the complete characterisation of $C(\gamma)$.

It could be possible that the maxmal B-invariant hypersurface H in the complement of $C(\gamma)$ is a lift, i.e., of the form $H = \pi_+^{-1}(H_+)$ (or $H = \pi_-^{-1}(H_-)$) with respect to the standard projection $\pi_+ : G^{\mathbb{C}}/K^{\mathbb{C}} \to X_+$ (or $\pi_- : G^{\mathbb{C}}/K^{\mathbb{C}} \to X_-$). Here, H_+ (resp. H_-) is the unique B-invariant hypersurface in X_+ (resp. X_-).

Distinguishing the case where the B-invariant hypersurface H is not lift and the case where H is a lift from either X_+ or from X_-, we prove our main results in Section 7 for non-closed G-orbits. The case of the closed G-orbit is a special case and is considered seperately.

If the B-invariant hypersurface H is not a lift, then the domain Ω_H, defined as the connected component containing the base point in $\Omega := G^{\mathbb{C}}/K^{\mathbb{C}}$ of the set $\Omega \setminus \bigcup_{k \in K}(kH)$, agrees with the universal domain Ω_{AG} ([FH]). This together with certain known results about the cycle spaces of open orbits ([GM], [HW1], [WZ]), leads to the proof of our main result in this situation.

If the B-invariant hypersurface H is a lift, then the base cycle $C_0 = \mathrm{c}\ell(\kappa)$ is P_+- (or P_--) invariant. Consequently, the orbit $G^{\mathbb{C}}.C_0$ in the cycle space $C_q(Z)$ is either the symmetric space X_+ or X_-. The cycle space $C(\gamma)$ in this case is just a lift of the domain Ω_{H_+} (or Ω_{H_-}).

The closed G-orbit $\gamma_{c\ell}$ is special in the sense that the Schubert slices in this case are just points. Consequently, Schubert slice intersection methods are not helpful. The case of the cycle space $C(\gamma_{c\ell})$ of the unique closed G orbit $\gamma_{c\ell}$ is handled in Section 8. Here, the boundary of the dual open $K^{\mathbb{C}}$-orbit κ_{op}, is decomposed into irreducible components $\mathrm{bd}(\kappa_{op}) = Z \setminus \kappa_{op} = A_1 \cup \ldots \cup A_k$ and it is shown that in each A_j there is a unique $K^{\mathbb{C}}$-orbit κ_j with dual G-orbit γ_j such that $\mathrm{c}\ell(\gamma_j) = \gamma_j \dot{\cup} \gamma_{c\ell}$. This leads to the inclusion $C(\gamma_{c\ell}) \subset C(\gamma_j)$. By showing that the dual open $K^{\mathbb{C}}$-orbit κ_{op} is neither P_+- nor P_--invariant, we obtain the equality $C(\gamma_{c\ell}) = \Omega_{AG}$ proving the main result for closed orbits as well.

In conclusion therefore, it is proven that except in a few explicit cases, the cycle space $C(\gamma)$ associated to any G-orbit γ in any $G^{\mathbb{C}}$-flag manifold $Z = G^{\mathbb{C}}/Q$ agrees with the universal domain Ω_{AG}. The only exception occurs when the real form G is of Hermitian type and the base cycle $\mathrm{c}\ell(\kappa)$ is P_+- (or

P_--) invariant. Then the cycle space $C(\gamma)$ associated to a nonclosed G-orbit γ is the bounded symmetric domain $\mathcal{B}$ (or its complex conjugate $\bar{\mathcal{B}}$).

I heartily thank my surpervisor Prof. Dr. Dr.h.c. A.T. Huckleberry for introducing me to this area of research and for always finding time to direct me in between tight schedules. My thanks also go to Prof. Dr. P. Heinzner and all the members of the department of complex analysis at the Ruhr Universität Bochum for their support throughout my Ph.D. programe. Finally, I would like to express my appriciation to the Deutsche Akademische Austauschdienst (DAAD) for supporting this work.

2 Preliminaries and Notations

2.1 Notations

Let G be a non-compact semi-simple Lie group which is embedded in its complexification $G^{\mathbb{C}}$ and let $Z = G^{\mathbb{C}}/Q$ be a $G^{\mathbb{C}}$-flag manifold, i.e., a compact, homogeneous, algebraic, rational $G^{\mathbb{C}}$-manifold. Here Q is a parabolic subgroup in the sense that it contains a Borel subgroup. Observe that G acts naturally on every flag manifold $Z = G^{\mathbb{C}}/Q$. The semi-simple Lie group G decomposes as a finite direct product of simple groups. This leads to a decomposition of each flag manifold $Z = Z_1 \times ... \times Z_k$ as a finite direct product with irreducible factors $Z_i = G_i^{\mathbb{C}}/Q_i$, where for each i, $Q_i := Q \cap G_i$ is a parabolic subgroup of the complexification $G_i^{\mathbb{C}}$ of the simple factors G_i. Thus a G-orbit in Z is a product of G_i-orbits in the corresponding factors Z_i. Consequently, in the sequel we will assume without loss of generality that G is simple. The necessary adjustments for the semi-simple case are straight-forward.

Fix a Cartan involution θ of G, and extend it as usual (holomorphically) to $G^{\mathbb{C}}$. The fixed point set $K := G^\theta$ is a maximal compactly embedded subgroup of G and $K^{\mathbb{C}} := (G^{\mathbb{C}})^\theta$ is its complexification. The compact group K as well as its complexification $K^{\mathbb{C}}$ also act naturally on Z and further, G/K is a negatively curved Riemannian symmetric space embedded in $G^{\mathbb{C}}/K^{\mathbb{C}}$.

Let $Orb_Z(G)$ (resp. $Orb_Z(K^{\mathbb{C}})$) denote the set of G-orbits (resp. $K^{\mathbb{C}}$-orbits) in Z. It is known that these sets are finite ([W1]). Since the G-action is algebraic, it follows that there is at least one open G-orbit. In fact, the

maximal-dimensional G-orbits are open while the minimal-dimensional G-orbits are closed. Moreover, there is only one closed G-orbit in Z.

There exists a duality relation between G-orbits and $K^{\mathbb{C}}$-orbits in any flag manifold Z. We will use the following slightly modified version of this duality relationship: *If $\kappa \in Orb_Z(K^{\mathbb{C}})$ and $\gamma \in Orb_Z(G)$, then (κ, γ) is said to be a dual pair if $\kappa \cap \gamma$ is non-empty and compact.*

If γ is an open G-orbit, however, then κ being dual to γ is equivalent to $\kappa \subset \gamma$. In ([W1]) it is shown that every open G-orbit contains a unique compact $K^{\mathbb{C}}$-orbit, i.e., duality at the level of open G-orbits.

Duality between G- and $K^{\mathbb{C}}$-orbits in Z is extended in ([M], see also [BL] and [MUV]) to the case of all orbits: *For every $\gamma \in Orb_Z(G)$ there exists a unique $\kappa \in Orb_Z(K^{\mathbb{C}})$ such that (γ, κ) is a dual pair and vice versa.*
Furthermore, if (γ, κ) is a dual pair then the intersection $\kappa \cap \gamma$ is transversal at each of its points and consists of exactly one K-orbit.

Let D be an open G-orbit in the flag manifold Z and C_0 the dual $K^{\mathbb{C}}$-orbit. Since C_0 is compact and contained in D, it defines a point in the space of q-dimensional compact cycles $\mathcal{C}_q(D)$, where $q := dim_{\mathbb{C}} C_0$. By associating $g \in G^{\mathbb{C}}$ to the cycle $g(C_0)$, the connected component $\Omega_W(D)$ of the set

$$\{g \in G^{\mathbb{C}} : g(C_0) \subset D\}$$

can be regarded as a family of q-dimensional cycles. Of course $\Omega_W(D)$ is invariant by the $K^{\mathbb{C}}$ action on $G^{\mathbb{C}}$ on the right and therefore, we often regard it as being in the affine homogenous space $G^{\mathbb{C}}/K^{\mathbb{C}}$.

Fix an open G-orbit D in $Z = G^{\mathbb{C}}/Q$, a base cycle C_0 in D and let $\Omega := G^{\mathbb{C}}.C_0$ denote the corresponding orbit in the cycle space $\mathcal{C}_q(D)$. The cycle space $\Omega_W(D)$ associated to open G-orbits in any flag manifold Z has been completely characterized ([FH]). The following result was proved:

Theorem 2.1. *([FH]). If Ω is compact, then either $\Omega_W(D)$ consists of a single point or G is Hermitian and $\Omega_W(D)$ is either the associated bounded symmetric domin $\mathcal{B}$ or its complex conjugate $\bar{\mathcal{B}}$. If Ω is non-compact, then regarding $\Omega_W(D)$ as a domain in $G^{\mathbb{C}}/K^{\mathbb{C}}$, it follows that $\Omega_W(D)$ agrees with the domain Ω_{AG} for every G-orbit in every $G^{\mathbb{C}}$-flag manifold $Z = G^{\mathbb{C}}/Q$.*

In terms of roots, the domain Ω_{AG} admits the following description. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of $Lie(G)$, with respect to a compact

real form $\mathfrak{g}_u$ of $\mathfrak{g}$. Let $\mathfrak{a} \subset \mathfrak{p}$ be an abelian subalgebra which is maximal with respect to the condition of being contained in $\mathfrak{p}$ and Φ a system of roots on $\mathfrak{a}$. This gives rise to an Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$, of $\mathfrak{g}$. For α a root of $\mathfrak{a}$, let $H_\alpha := \{\xi \in \mathfrak{a} : \alpha(\xi) = \frac{\pi}{2}\}$ and define ω_{AG} as the connected component containing $0 \in \mathfrak{a}$ of the set which is obtained by removing from $\mathfrak{a}$ the union of all the affine hyperplanes H_α as α runs through the set of roots. That is,

$$\omega_{AG} = (\mathfrak{a} \setminus (\bigcup_{\alpha\in\Phi} H_\alpha))^o = \bigcap_{\alpha\in\Phi} \{\xi \in \mathfrak{a} : |\alpha(\xi)| < \frac{\pi}{2}\}.$$

Definition 2.1. Let ω_{AG} be as above, then the domain Ω_{AG} is the open neigborhood of the Riemannian symmetric space G/K in $\Omega := G/K^{\mathbb{C}}$ given by

$$\Omega_{AG} = Gexp(i\omega_{AG}).x_0,$$

where $x_0 \in G^{\mathbb{C}}/K^{\mathbb{C}}$ is the base point.

This domain Ω_{AG} has the property that the G-action on it is proper ([AG]). For a survey of this and other basic properties of Ω_{AG} see ([HW2]).

Our main aim in this work is to define and characterize in general the cycle space associated to any G-orbit γ in any $G^{\mathbb{C}}$-flag manifold Z. With duality between G and $K^{\mathbb{C}}$-orbits in mind, a natural candidate for the cycle space associated to an arbitrary G-orbit γ would be the connected component $C(\gamma)$ of the set

$$\{g \in G^{\mathbb{C}} : g(\kappa) \cap \gamma \text{ is non-empty and compact}\},$$

for $(\gamma, \kappa) \in Orb_Z(G) \times Orb_Z(K^{\mathbb{C}})$ a dual pair. This set was introduced in ([GM]) and it was conjectured that the intersection of all such sets for all $K^{\mathbb{C}}$-orbits in all $G^{\mathbb{C}}$-flag manifolds $G^{\mathbb{C}}/Q$ agrees with the universal domain Ω_{AG}. Furthermore, this conjecture was shown (in the same paper) to be true for classical and exceptional Hermitian groups by case-by-case considerations.

However, it is not quite clear what happens at the boundary of γ. Thus we can not apriori say if the set $C(\gamma)$ is non-empty or not. We will begin by investigating the intersection of $c\ell(\kappa)$ wih the boundary of γ. This information will lead us to understand boundary behaviour of *cycles* and hence a suitable definition of the cycle space $C(\gamma)$ that holds for open as well as for non-open G-orbits in any flag manifold Z in general.

Let us now put together some preparatory results. We will closely follow the notation in ([HW1]).

Let B denote a Borel subgroup of $G^{\mathbb{C}}$ which contains the factor AN of an Iwasawa decomposition $G = KAN$ of G. Such a Borel subgroup is called an *"Iwasawa-Borel"* subgroup of $G^{\mathbb{C}}$.

Given B, an Iwasawa-Borel subgroup of $G^{\mathbb{C}}$, the closure $S = \mathrm{c}\ell(O) \subset Z$ of a B-orbit O in Z is referred to as an Iwasawa- Schubert variety. We set $Y := S \setminus O$. For a fixed Iwasawa-Borel subgroup B, let $\mathcal{S}$ denote the set of all Schubert varieties and define for every $\kappa \in Orb_Z(K^{\mathbb{C}})$ the set

$$\mathcal{S}_\kappa := \{S \in \mathcal{S} : S \cap \mathrm{c}\ell(\kappa) \neq \emptyset\}.$$

This set is non-empty since the set of Schubert varieties generate the integral homology of Z.

Given a dual pair $(\gamma,\kappa) \in Orb_Z(G) \times Orb_Z(K^{\mathbb{C}})$, a Borel subgroup B of $G^{\mathbb{C}}$ which contains the factor AN of an Iwasawa-decomposition $G = KAN$ of G, a B Schubert variety $S \in \mathcal{S}_\kappa$ and an intersection point $z_0 \in \kappa \cap S$, we refer to $\Sigma = AN.z_0$ as the associated *"Schubert slice"*.

In what follows, fix an Iwasawa decomposition $G = KAN$ and an Iwasawa-Borel subgroup B containing the factor AN. Set $C_0 := \mathrm{c}\ell(\kappa)$, the closure of the $K^{\mathbb{C}}$-orbit κ, dual to $\gamma \in Orb_Z(G)$.

Lemma 2.2. *If $p \in \Sigma \cap C_0$, then the tangent space to γ at p decomposes as a direct sum*

$$T_p\gamma = T_p(C_0) \oplus T_p(\Sigma).$$

Proof. From the Iwasawa decomposition $G = KAN$, it follows in particular, that $T_p\gamma = T_p(C_0) + T_p(AN.p)$. To see that this sum is direct, it just suffices to count dimensions noting that the orbit $AN.p$ is contained in S. $\square$

Proposition 2.3. *Let (κ, γ) be a dual pair and $S \in \mathcal{S}_\kappa$. Then*

1. *Any Schubert slice Σ is open in S*
2. *At each of its intersection points S is transversal to $\cap C_0$ in γ.*
3. *The intersection $S \cap C_0$ is finite and contained in $\mathcal{O}$. Moreover, $S \cap \gamma$ is a finite disjoint union $\dot\cup_{j=1}^{d} \Sigma_j$ of Schubert slices.*

Proof. Since $dim_{\mathbb{C}}\Sigma + dim_{\mathbb{C}}C_0 = dim_{\mathbb{C}}\gamma$, it follows that $dim_{\mathbb{R}}S = dim_{\mathbb{R}}AN.p$ and as a consequence, $AN.p$ is open in S. Tranversality now follows since we have the direct sum decomposition $T_p\gamma = T_p(C_0) \oplus T_pS$ of the tangent space.

Any component of $S \cap \gamma$ is AN-invariant and since every AN-orbit in γ intersects C_0, it follows that such an orbit is open in $S \cap \gamma$. Consequently, such a component is a Schubert slice. It follows that $S \cap C_0 = \{p_1, \ldots, p_d\}$, and $\Sigma_j := AN.p_j$ are the corresponding Schubert slices through p_j. Hence $S \cap C_0 = S \cap \mathcal{O} \subset \mathcal{O}$ is just the disjoint union of the Schubert slices Σ_j.

□

The following result is implicit in ([HW1], see Section 5). There it was only proven that the intersection $\Sigma \cap C_0$ is finite.

Proposition 2.4. *The intersection $\Sigma \cap C_0$ consists of exactly one point for any Schubert slice Σ.*

Proof. Let Σ be a Schubert slice through $p \in \Sigma \cap C_0$. Suppose Σ intersects C_0 in another point p'. Then since C_0 is a K-orbit, there exists $k \in K$ such that $k.p' = p$. Since $p \in \Sigma$, there exists $an \in AN$ such that $(an).p = p'$. It then follows that kan belongs to the G-isotropy subgroup at p.

The map $\alpha : K_p \times (AN)_p \to G_p$ defined by multiplication $(k, an) \mapsto kan$ is a diffeomorphism ([HW1]) onto a number of components of G_p. However, since $\gamma \cap \kappa$ is a strong deformation retract of γ, it follows that G_p/K_p is connected and consequently, α is surjective. It therefore follows that k belongs to K_p, the K-isotropy subgroup at p. Thus $k.p' = p' = p$. □

2.2 Definition of the cycle Space

Our aim here is to give a suitable definition of the cycle space associated to a G-orbit in any flag manifold $Z = G^{\mathbb{C}}/Q$. We will need the following result in the sequel.

Proposition 2.5. *[HW1]. Let (κ, γ) be a dual pair and $S \in \mathcal{S}_\kappa$. Then*

1. *$S \cap c\ell(\kappa) \subset \kappa \cap \gamma$.*

2. *The map $K \times c\ell(\Sigma) \to c\ell(\gamma)$, given by $(k, z) \mapsto k(z)$, is surjective, that is $K.c\ell(\Sigma) = c\ell(\gamma)$.*

Corollary 2.6. *Every $p \in c\ell(\gamma)$ is contained in some Schubert variety $S \in \mathcal{S}_\kappa$.*

Proof. Let B be an Iwasawa-Borel subgroup containing the factor AN of some given Iwasawa decomposition $G = KAN$ of G. Let z_0 be the base point of γ so that $z_0 \in \kappa \cap \gamma$ and $S \in \mathcal{S}_\kappa$ be an Iwasawa-Schubert variety through z_0, that is , $S = \mathrm{c}\ell(B.z_0)$. Furthermore, let $\Sigma = AN.z_0$ be a Schubert slice through z_0. Suppose that $p \in \mathrm{c}\ell(\gamma)$, then by Prop. 2.5, there exists $k \in K$ such that $p \in k.\mathrm{c}\ell(\Sigma)$. The K-conjugate kBk^{-1} of B contains the conjugate $kANk^{-1}$ of AN and as a consequence, $p \in k.\mathrm{c}\ell(kANk^{-1}.z_0) = \mathrm{c}\ell((kAk^{-1})(kNk^{-1})).z_0 = \mathrm{c}\ell(\tilde{A}\tilde{N}.z_0) \subset \mathrm{c}\ell(\tilde{B}.z_0) = \tilde{S}$. Here $\tilde{S}$ is another Schubert variety which is the closure of the orbit $\tilde{\mathcal{O}}$ of the Iwasawa-Borel subgroup $\tilde{B}$ containing the factor $\tilde{A}\tilde{N}$ of some other Iwasawa decomposition of G. □

Lemma 2.7. $c\ell(\kappa) \cap c\ell(\gamma) = \kappa \cap \gamma$.

Proof. Of course we have that $\kappa \cap \gamma \subset \mathrm{c}\ell(\kappa) \cap \mathrm{c}\ell(\gamma)$ and so it is sufficient to prove the opposite inclusion. Suppose $p \in \mathrm{c}\ell(\kappa) \cap \mathrm{c}\ell(\gamma)$, then by Cor. 2.6, there is some S containing the point p, that is $p \in \mathrm{c}\ell(\kappa) \cap S$, and this intersection is contained in $\kappa \cap \gamma$ by the first part of Prop. 2.5.

□

Lemma 2.8. *If Y is any compact set in Z with $Y \cap c\ell(\gamma) \subset \gamma$, then there is an open neighborhood U of the Id in $G^{\mathbb{C}}$ such that $g(Y)$ has the same property.*

Proof. Let $d : Z \times Z \to \mathbb{R}^{\geq 0}$ be any distance function on Z. Since $Y \cap \mathrm{c}\ell(\gamma) \subset \gamma$, and Y is compact, it follows that the distance between Y and $\mathrm{bd}(\gamma)$ is positive, that is, $d(Y, \mathrm{bd}(\gamma)) > 0$. Furthermore, the function $\beta : G^{\mathbb{C}} \to \mathbb{R}^{\geq 0}$, given by $g \mapsto d(g(Y), \mathrm{bd}(\gamma))$ is continuous. Since $\beta(Id) > 0$, it follows that there exists a neigborhood U of the identity $Id \in G^{\mathbb{C}}$ such that $d(gY, \mathrm{bd}(\gamma)) > 0$ for $g \in U$. Consequently, $g(Y) \cap \mathrm{c}\ell(\gamma) \subset \gamma$, for any $g \in U$.

□

Corollary 2.9. *The identity is an interior point of the set*

$$C := \{g \in G^{\mathbb{C}} : g(\kappa) \cap \gamma \textit{ is non-empty and compact}\}.$$

Proof. We have by Lemma 2.7, that $c\ell(\kappa) \cap c\ell(\gamma) = \kappa \cap \gamma$ which is contained in γ. Since $c\ell(\kappa)$ is compact, Lemma 2.8 implies that there exists a neigborhood U of the identity in $G^{\mathbb{C}}$ such that $g(c\ell(\kappa)) \cap c\ell(\gamma) = g(\kappa) \cap \gamma \subset \gamma$ for $g \in U$. Since γ and κ are dual, the intersection $\gamma \cap \kappa$ is tranversal. Consequently, for g in a possibly smaller neigborhood as U, the intersection $g(\kappa) \cap \gamma$ remains transversal and therefore non-empty. It now follows that the intersection $g(c\ell(\kappa)) \cap c\ell(\gamma)$ for $g \in U$ is non-empty. This implies that the identity is an interior point of the set C. □

Definition 2.2. Let $(\gamma, \kappa) \in Orb_Z(G) \times Orb_Z(K^{\mathbb{C}})$ be a dual pair. The *cycle space* $C(\gamma)$ associated to a G-orbit γ is the connected component containing the identity of the interior of the set

$$C = \{g \in G^{\mathbb{C}} : g(\kappa) \cap \gamma \text{ is non-empty and compact}\}.$$

It is now clear that $C(\gamma)$ is a non-empty open subset of $G^{\mathbb{C}}$ since the identity belongs to the set C. Furthermore, if $\gamma = D$ is an open G-orbit in Z, then $C(\gamma)$ agrees with the cycle domain $\Omega_W(D)$ introduced in ([WeW]).

Of course elements of $C(\gamma)$ are transformations and not cycles in the sense of points in the cycle space $\mathcal{C}_q(Z)$. There will be occasions, however, where we really want to think of an element of $C(\gamma)$ as a cycle in the latter sense. For this we recall that the $G^{\mathbb{C}}$-action on $\mathcal{C}_q(Z)$ is algebraic and therefore the orbit $G^{\mathbb{C}}.C_0$ can be identified with the algebraic homogeneous space $G^{\mathbb{C}}/G^{\mathbb{C}}_{C_0}$.

It follows that the transformation group variant $C(\gamma)$ is invariant under right-multiplication by $G^{\mathbb{C}}_{C_0}$. Thus, if we wish to think of cycles as being in $\mathcal{C}_q(Z)$, we replace $C(\gamma)$ by $C(\gamma)/G^{\mathbb{C}}_{C_0}$.

Now, the isotropy subgroup $G^{\mathbb{C}}_{C_0}$ always contains $K^{\mathbb{C}}$. In fact, if G is not of Hermitian type, then $K^{\mathbb{C}}$ is maximal in $G^{\mathbb{C}}$ in the sense that the only proper subgroups which contain it are finite extensions $\tilde{K}^{\mathbb{C}}$. Thus in the nonhermitian case $G^{\mathbb{C}}_{C_0}$ is at most a finite extension of $K^{\mathbb{C}}$.

In the Hermition case $K^{\mathbb{C}}$ is properly contained in the parabolic subgroup P_+ or P_-, where $G^{\mathbb{C}}/P_+$ and $G^{\mathbb{C}}/P_-$ are the associated compact Hermitian symmetric spaces. Thus it is quite possible that $G^{\mathbb{C}}_{C_0}$ is one of these subgroups. For example, if x_+ is the base point in $G^{\mathbb{C}}/P_+$ and $\gamma = G.x_+$ is its (open) orbit, then κ is just the base point and $G^{\mathbb{C}}_{C_0} = P_+$.

In the sequel we will sometimes regard $C(\gamma)$ as being in $\mathcal{C}_q(Z)$, i.e., in $G^{\mathbb{C}}/G^{\mathbb{C}}_{C_0}$, where $G^{\mathbb{C}}_{C_0}$ is one of the groups described above. On occasion,

either by going to a finite cover or pulling back by one of the fibrations $G^{\mathbb{C}}/K^{\mathbb{C}} \to G^{\mathbb{C}}/P_{\pm}$ we will regard it in $G^{\mathbb{C}}/K^{\mathbb{C}}$.

Our work here makes use of results and methods from ([BK], [FH], [HW1], [HW2]) which together with knowledge of the intersection of cycle domains for the open orbits in $G^{\mathbb{C}}/B$ ([GM]), implies the inclusion $\Omega_{AG} \subset C(\gamma)$ for all $\gamma \in Orb_Z(G)$. This inclusion plays an important role in our proofs.

3 Characterization of $C(\gamma)$ by Schubert slices

Throughout this section, C_0 will denote $c\ell(\kappa)$ where $(\gamma, \kappa) \in Orb_Z(G) \times Orb_Z(K^{\mathbb{C}})$ is a dual pair. Futhermore, we assume that the G-orbit γ under consideration is not closed. We give a characterization of the cycle space $C(\gamma)$ associated to γ by means of cycle intersection with Schubert slices. Our goal here is to prove the following

Proposition 3.1. *Let Σ be a Schubert slice and suppose that $\{g_n\}$ is a sequence in $C(\gamma)$ such that $g_n \to g$ with $g_n(c\ell(\kappa)) \cap \Sigma = \{p_n\}$ and p_n diverges in Σ. Then $g \notin C(\gamma)$.*

In order to prove the above result, we need some preparations.

Lemma 3.2. *For all $g \in C(\gamma)$ the number of points in the intersection $g(\kappa) \cap \Sigma$ is bounded by the intersection number $[S].[c\ell(\kappa)]$.*

Proof. Since Σ can be regarded as a domain in $\mathcal{O} \cong \mathbb{C}^m$ and $g(\kappa) \cap \gamma$ is compact, it follows from the maximum principle that $g(\kappa) \cap \Sigma$ is finite and of course it is then bounded by the intersection number $[S].[c\ell(\kappa)]$. $\square$

Define a subset $\mathcal{I}$ of $G^{\mathbb{C}}$ as the connected component containing the identity of the interior of the set

$$\{g \in G^{\mathbb{C}} : |g(\kappa) \cap \Sigma| = 1 \text{ for all } \Sigma\}.$$

Observe that since $c\ell(\kappa) \cap c\ell(\gamma) = \kappa \cap \gamma$, and $|\kappa \cap \Sigma| = 1$ for all Σ, it follows that $g \in \mathcal{I}$ for g sufficiently close to the identity. Thus $\mathcal{I}$ is an open subset of $G^{\mathbb{C}}$ containing the identity.

In the definition of $\mathcal{I}$ above, *for all* Σ means for all choices of the maximal compact group K and all Iwasawa factors AN, i.e., all Schubert slices which arise by G-conjugation of those Σ which are connected components of $S \cap \gamma$ for a fixed $S \in \mathcal{S}_\kappa$

Lemma 3.3. *For all $g \in \mathcal{I}$, the intersection $g(\kappa) \cap \gamma$ is transversal.*

Proof. Let d denote the intersection number $[S].[\mathrm{c}\ell(\kappa)]$, then the base cycle $\mathrm{c}\ell(\kappa)$ intersects $\mathcal{O}$ in exactly d points. Furthermore, $\mathcal{O} \cap \gamma$ is a disjoint union of Schubert slices $\Sigma_1, \ldots, \Sigma_d$ with one intersection point in each slice. Thus if $g \in \mathcal{I}$, then $g(\mathrm{c}\ell(\kappa))$ intersects each Σ_i in exactly one point as well. If any of such intersection point were not transversal, then the intersection number would be too big. □

The following is a consequence of the above Lemma.

Corollary 3.4. *The intersection $M_g = g(\kappa) \cap \gamma$ is a connected compact manifold for all $g \in \mathcal{I}$.*

Proposition 3.5. *$bd(\mathcal{I}) \cap C(\gamma) = \emptyset$.*

Proof. Assume by contradiction that $g \in \mathrm{bd}(\mathcal{I}) \cap C(\gamma)$. Let $\{g_n\}$ be a sequence in $I \cap C(\gamma)$ with $g_n \to g$. Then by Cor. 3.4, $M_n := g_n(\kappa) \cap \gamma$ is a sequence of compact connected manifolds in γ.

Let $\tilde{M}$ denote the limiting set, $\tilde{M} := \lim M_n$. It follows that $\tilde{M}$ is a connected closed subset of $\mathrm{c}\ell(\gamma)$.

Now,

$$\tilde{M} \subset g(\mathrm{c}\ell(\kappa)) \cap \mathrm{c}\ell(\gamma) =: A \,\dot{\cup}\, E,$$

where

$$A := (g(\mathrm{bd}(\kappa)) \cap \mathrm{c}\ell(\gamma)) \cup (g(\kappa) \cap \mathrm{bd}(\gamma)) := A_1 \,\dot{\cup}\, A_2$$

and

$$E := g(\kappa) \cap \gamma.$$

The set A is closed, because A_1 is the intersection of two closed sets, and a sequence in A_2 which converges in Z will either converge to a point of A_2 or A_1.

Since we have assumed that $g \in C(\gamma)$, it follows that E is compact. Thus

$$\tilde{M} = A \,\dot{\cup}\, E$$

is a decomposition of $\tilde{M}$ into disjoint open subsets of $\tilde{M}$. Since $\tilde{M}$ is connected and $\tilde{M} \cap A \neq \emptyset$, we conclude that $\tilde{M} \subset A$.

Consequently, for every relatively compact open neighborhood U of a point $p \in \gamma$, there exists a positive integer $N = N(U)$ such that $g_n(\kappa) \cap U = \emptyset$ for all $n > N$.

Now we have assumed that $g \in C(\gamma)$, in particular that E is nonempty. Hence, for $p \in E$ we can consider an Iwasawa-Schubert variety $S = \mathcal{O} \dot{\cup} Y$ with $p \in \mathcal{O}$. Since E is compact, the complex analytic set $g(\mathrm{c}\ell(\kappa)) \cap S$ must contain p as an isolated point. Thus, for Σ a Schubert slice through p, the intersection $g(\kappa) \cap \Sigma$ is isolated at p and as a consequence, $g_n(\kappa)$ must have nonempty intersection with any open neighborhood $U = U(p)$ of p if n is sufficiently large. This is contrary to the above statement and therefore $g \notin C(\gamma)$.

□

Proposition 3.6. *$C(\gamma) \subset \mathcal{I}$.*

Proof. By the openess of $\mathcal{I}$ and the fact that $\mathrm{bd}(\mathcal{I}) \cap C(\gamma) = \emptyset$ (see, Prop. 3.5 above), it follows in particular that $\mathcal{I} \subset C(\gamma)$. □

Proof of Proposition 3.1. Again we argue by contradiction. Suppose that $g \in C(\gamma)$. Then by Prop. 3.6, $g(\mathrm{c}\ell(\kappa)) \cap \Sigma$ consists of exactly one point, say q. Since p_n diverges in Σ, we may assume that $p_n \to p \in \mathrm{c}\ell(\Sigma) \setminus \Sigma$.

Now by definition $C(\gamma)$ is open. Thus there exists a small $h \in G^{\mathbb{C}}$ with hg still in $C(\gamma)$ and $hg(\kappa) \cap \Sigma$ containing points near p and q. Thus, $|hg(\kappa) \cap \Sigma| \geq 2$, in violation of $C(\gamma) \subset \mathcal{I}$. □

Corollary 3.7. *If $\{g_n\}$ is a sequence of cycles in $C(\gamma)$ such that $g_n \to g$ and $g_n(c\ell(\kappa)) \cap (\gamma \cap S)$ contains a sequence p_n which diverges in $\mathcal{O}$, then $g \notin C(\gamma)$.*

Proof. Since $\mathcal{O} \cap \gamma$ is a finite union of Schubert slices $\Sigma_1 \cup \ldots \cup \Sigma_k$ say, and the sequence $\{p_n\}$ diverges in $\mathcal{O}$, it follows that some Schubert slice contains infinitely many points of the sequence $\{p_n\}$. We may therefore assume that the sequence $\{p_n\}$ is contained in some fixed Schubert slice Σ. This implies that $p_n \to p \in \mathrm{c}\ell(\Sigma) \setminus \Sigma$, and it follows from Prop. 3.1 that $g \notin C(\gamma)$. □

4 The trace transform

As usual let Z denote an arbitrary flag manifold and z_0 the base point of a non-closed G-orbit γ. In this section we regard a cycle as being in $\mathcal{C}_q(Z)$.

For $(\gamma, \kappa) \in \mathrm{Orb}_Z(G) \times \mathrm{Orb}_Z(K^{\mathbb{C}})$ a dual pair, let $S \in S_\kappa$ be an Iwasawa Schubert variety throught the base point z_0. We recall that S is the closure of $\mathcal{O}$, the orbit of an Iwasawa-Borel subgroup B. That is, S is the disjoint union $S = \mathcal{O} \dot{\cup} Y$.

Let Ψ consist of all pairs (S, Y) which occur as above, then we refer to an element $\psi \in \Psi$ as datum for a *trace-transform* $\mathcal{T}_\psi$ (see definition below), with respect to ψ.

Define a subset of the space $\mathcal{C}_q(Z)$ of q-dimensional cycles in Z by

$$\Omega_\psi := \{C \in \mathcal{C}_q(Z) : C \cap Y = \emptyset\}.$$

Since the condition $C \cap Y \neq \emptyset$ defines a closed analytic subset in $\mathcal{C}_q(Z)$, it follows that Ω_ψ is Zariski open and dense in the irreducible component of $\mathcal{C}_q(Z)$ which contains the base cycle.

Let $\mathcal{M}(S)$ denote the ring of meromorphic functions on S and

$$\Gamma(S, \mathcal{O}(*Y)) := \{f \in \mathcal{M}(S) : \mathcal{P}(f) \subset Y\}$$

the subring of functions with polar set $\mathcal{P}$ contained in Y.

Definition 4.1. The trace-transfomation, $\mathcal{T}_\psi$, with respect to $\psi \in \Psi$ is defined by

$$\mathcal{T}_\psi : \Gamma(S, \mathcal{O}(*Y)) \to \mathcal{M}(\Omega_\psi),\ f \mapsto \mathcal{T}_\psi(f) := \sum_{p \in C \cap S} f(p).$$

When there is no ambiguity, we will just write $\mathcal{T}$ for $\mathcal{T}_\psi$.

Observe that regarded as a function, $\mathcal{T}_\psi(f) : \Omega_\psi \to \mathbb{C}$, the trace-transformation $\mathcal{T}_\psi(f)$ is infact a composition of two maps. Firstly, the intersection map $I : \Omega_\psi \to \mathcal{C}_q^0(Z)$, which associates to a cycle its intersection with S, and is holomorpic. Secondly, the trace map $\mathrm{trace}(f) : Sym_k(S \setminus Y) \to \mathbb{C}$ given by

$$\mathrm{trace}(f)(z_1, \ldots, z_k) = \sum_{j=i}^{k} f(z_j).$$

which is also holomorphic.
We need the following

Proposition 4.1. *[HSB]. Let Γ be a closed irreducible subspace of $\mathcal{C}_q(Z)$ such that $\Gamma \cap \Omega_\psi \neq \emptyset$. In particular, $\Gamma \cap \Omega_\psi$ is a dense, Zariski open set in Γ. Then $\mathcal{T}_\psi(f)$ is holomorphic on $\Gamma \cap \Omega_\psi$ and extends meromorphically to Γ.*

Proof. We have already seen above that $\mathcal{T}_\psi(f)$ is a composition of two maps, the intersection map and the trace map. Infact, the intersection map

$$I : \Gamma \cap \Omega_\psi \to \mathrm{Sym}^k(S \setminus Y)$$

is give by $C \mapsto I(C) = C \cap S$ (of course in the cycle sense), where $k \in \mathbb{N}$ depends on Γ. Since the projection map $\Gamma \times Sym_k(S) \times S \to \Gamma$ is proper and the set

$$\{(C, (z_1, \ldots, z_k), z) \in \Gamma \times Sym_k(S) \times S : z \in C \text{ and } z = z_i \text{ for some } i\}$$

is a closed analytic set in $\Gamma \times Sym_k(S) \times S$, we have that I extends to a meromorphic map $\tilde{I} : \Gamma \to Sym_k(T)$. The trace map on the other hand, $trace(f)$ map;

$$trace(f) : Sym^k(S \setminus Y) \to \mathbb{C}$$

given by $trace(f)(z_i, \ldots, z_k) = \sum_{j=1}^{k} f(z_j)$, is holormorphic and extends as a meromorphic function $\tilde{f} : \mathrm{Sym}^k(S) \to \mathbb{C}$. Indeed, in the neigborhood of a point $(z_1^0, \ldots, z_k^0) \in Sym_k(S)$, where some of the z_j^0 belong to Y, choose g holomorphic near $\{z_1^0\} \cup \ldots \cup \{z_k^0\}$ such that $g \equiv 1$ if $z_i^0 \notin Y$, then $g \times f$ is holomorphic. Set $G(z_1^0, \ldots, z_k^0) = \Pi_{j=1}^{k} g(z_j)$, if $(z_1, \ldots, z_k)$ is near $(z_1^0, \ldots, z_k^0)$, then

$$(G \times \mathrm{trace}(f)(z_1, \ldots, z_k) = \sum_{j=1}^{k} g(z_1) \ldots \widehat{g(z_i)} \ldots .g(z_k).(g \times f)(z_k),$$

(where $\widehat{g(z_i)}$ means omit this factor in the above sumation), is meromorphic on $Sym_k(S)$ near $(z_1^0, \ldots, z_k^0)$.

Consequently, trace(f) extends meromorphically on $Sym_k(S)$ and so $\mathcal{T}_\psi(f)$ is also meromorphic as a composition $trace(f) \circ I$ of two meromorphic maps on all of Γ. $\square$

Corollary 4.2. *The trace-transform $\mathcal{T}_\psi(f)$ extends meromorphically to the component of the space of q-dimensional cycles $\mathcal{C}_q(Z)$ which contains Ω.*

5 Embedding of the Schubert variety

Let $S \in \mathcal{S}_\kappa$ be a Schubert variety. It is our aim in this section to embed $S = \mathcal{O} \dot{\cup} Y$ into a projective space so that Y is the intersection of S with the hyperplane at infinity.

Let $L \to Z$ be any very ample line bundle on Z and let $L|_S \to S$ be its restriction to S. Without loss of generality, we may assume that $G^{\mathbb{C}}$ is simply connected. Then the line bundle $L \to Z$ is $G^{\mathbb{C}}$-homogeneous and consequently, the restriction to S is B-homogeneous as well. Let $\Gamma(L, Z)$, (resp. $\Gamma(L|_S, S)$) denote the finite dimensional vector spaces of holomorphic sections of the respective bundles. Then there exists a natural B-equivariant restriction map

$$r : \Gamma(L, Z) \to \Gamma(L|_S, S) \text{ given by } \sigma \mapsto \sigma|_S.$$

Since $L \to Z$ is very ample, it yields a holomorphic B-equivariant embedding onto its image;

$$\varphi : S \to \mathbb{P}(Im(r)^*) \text{ given by } s \mapsto H_s := \{\sigma \in Im(r) : \sigma(s) = 0\}.$$

Moerover, if $((\sigma_0, \ldots, \sigma_m))$ is a basis for $Im(r)$, then φ may be defined in coordinates by the following map

$$S \mapsto \mathbb{P}_m(\mathbb{C}), s \mapsto \varphi(s) = [\sigma_0(s), \ldots, \sigma_m(s)].$$

By the Borel fixed point theorem, B has an eigenvector $r(\sigma_B)$ in $Im(r) \setminus \{0\}$. Suppose $\sigma_0 := r(\sigma_B)$ is a B-eigenvector and let $H := \{\sigma_0 = 0\}$, then H is B-invariant. Using the above notation, we first note the following

Proposition 5.1. *$H \subset Y$.*

Proof. Since H is B-invariant, it follows that $S \setminus H$ is also B-invariant. Since both $\mathcal{O}$ and $S \setminus H$ are Zariski dense, their intersection is nonempty. Thus for $s \in \mathcal{O} \cap (S \setminus H)$, it follows that the B-orbit $\mathcal{O} = B.s$ is contained in $S \setminus H$. This implies that H is contained in the complement of $\mathcal{O}$. $\square$

With respect to the above embedding φ therefore, the open B-orbit $\mathcal{O}$ is embedded in $\mathbb{C}^m$ as follows;

$$\mathcal{O} \to \mathbb{C}^m, \; s \mapsto (\frac{\sigma_1}{\sigma_0}(s), \ldots, \frac{\sigma_m}{\sigma_0}(s)),$$

and as such, it could be possible that some components of Y are not in H. However, we proceed to show that infact $H = Y$. For this, we need the following

Lemma 5.2. *If U is a unipotent group acting algebraically as a group of affine transformations on $\mathbb{C}^n$, then every orbit is closed.*

Proof. Let $(w_1, \ldots, w_m)$ represent coordinates for $\mathbb{C}^m$. Assume that the action is in upper triangular form and that the coordinates are arranged so that the projection $\pi : \mathbb{C}^m \to \mathbb{C}^{m-1}$ given by $(w_1, \ldots, w_m) \mapsto (w_2, \ldots, w_m)$ is U-equivariant.

Our aim is to show that for $p \in \mathbb{C}^m$, the orbit $U.p$ is closed. Let $q := \pi(p)$ and assume inductively that $U.q$ is closed in $\mathbb{C}^{m-1}$. Since $U.q$ is closed, it follows that $\pi(c\ell(U.p)) = U.q$.

Suppose firstly that $U(p)$ and $U(q)$ have the same dimension. Then if $U(p)$ were not closed, it would have an orbit of lower dimension on its boundary which would be smaller than $U(q)$. But this is impossible, because it would be mapped onto $U(q)$.

On the other hand, suppose that $U(p)$ is dimension-theoretically larger than $U(q)$, that is one dimension bigger. Now every fiber of $U(p) \to U(q)$ is a copy of $\mathbb{C}$ which is open in the π-fiber. If $U(p)$ were not closed, then its closure in the π-fiber would contain an additional point. Consequently, this closure would be $\mathbb{P}_1$. But the π-fiber is a copy of $\mathbb{C}$ which certainly does not contain a $\mathbb{P}_1$. Hence every fiber of $U(p) \to U(q)$ is a π-fiber and as a consequence, $U(p)$ is the π-preimage of the closed set $U(q)$ and thus is also closed.

Since every U-orbit on $\mathbb{C}$ is clearly closed, we conclude by the induction hypothesis that all U-orbits in $\mathbb{C}^m$ are closed. $\square$

Now let U denote the unipotent radical of the Iwasawa-Borel subgroup B. We put all the above results together in the following

Proposition 5.3. *Let $S \in \mathcal{S}_\kappa$ be a Schubert variety and $L \to Z$ any very ample line bundle on Z and suppose*

$$r : \Gamma(L, Z) \to \Gamma(L|_S, S) \text{ given by } \sigma \mapsto \sigma|_S$$

denotes the canonical restriction map. Furthermore, let the embedding of S in to a projective space be given in coordinates by

$$S \mapsto \mathbb{P}_m(\mathbb{C}), s \mapsto [\sigma_0(s), \ldots, \sigma_m(s)]$$

where $((\sigma_0, \ldots, \sigma_m))$ denotes a basis for $Im(r)$. If σ_0 is chosen to be a B-eigenvector, then the B-invariant hyperplane $\{\sigma = 0\}$ is the complement Y of the B-orbit $\mathcal{O}$.

Proof. Since U is the unipotent radical of the Iwasawa-Borel subgroup B, it acts transitively on the B-orbit $\mathcal{O}$. Now the line bundle $L \to Z$ is very ample and defines a B-equivariant embedding $\varphi : S \to \mathbb{P}(Im(r)^*)$ such that the zero set of σ_0, that is $H = \{\sigma_0 = 0\}$, is contained in the complement of the open B-orbit $\mathcal{O}$. It follows that

$$S \setminus \{\sigma_0 = 0\} \to \mathbb{C}^m \text{ given by } s \mapsto (\frac{\sigma_1}{\sigma_0}(s), \ldots, \frac{\sigma_m}{\sigma_0}(s))$$

is a U-equivariant embedding of $S \setminus H$ into $\mathbb{C}^m$. By Lemma 5.2, it follows that the U-orbit in $S \setminus H$ is both open and closed in $\mathbb{C}^m$. Consequently, the U-action on $S \setminus H$ is transitive and therefore we conclude that $H = Y$. $\square$

Since the B-orbit $\mathcal{O}$ is algebraically isomorphic to $\mathbb{C}^{m(\mathcal{O})}$, we will later on apply the above result to show that any sequence in $\mathcal{O}$ that converges to a point in Y, converges to infinity in $\mathbb{C}^{m(\mathcal{O})}$. More precisely, we have the following

Corollary 5.4. *If a sequence $\{p_n\} \in \mathcal{O}$ converges to a point $p \in Y$ then with respect to the above embedding, it converges to infinity in $\mathbb{C}^m$.*

Proof. Let $[z_0 : \ldots : z_m]$ represent homogeneous coordinates for $\mathbb{P}_m(\mathbb{C})$ such that the unipotent group U fixes the coordinate z_0. The affine action of U is given by the restriction of its action on $\mathbb{P}_m(\mathbb{C})$ to the complement of $\{z_0 = 0\}$ with affine coordinates $(w_1, \ldots, w_m)$ where $w_j := \frac{z_j}{z_0}$ for $j = 1, \ldots, m$. Since the embedding of $S \setminus H$ into $\mathbb{C}^m$ is U-equivariant and $H = Y$ by Prop. 5.3, it follows that if $p_n \in \mathcal{O}$ converges to a point $p \in Y$ then it converges to ∞ in $\mathbb{C}^m$. $\square$

6 Incidence variety I_Y

As usual, let $(\gamma, \kappa) \in Orb_Z(G) \times Orb_Z(K^{\mathbb{C}})$ be a dual pair and regard the cycle space $C(\gamma)$ as being contained in the group $G^{\mathbb{C}}$. In what follows, $p_0 \in \kappa \cap \gamma$ will denote the base point.

For an Iwasawa-Schubert variety $S \in \mathcal{S}_\kappa$ defined by an Iwasawa-Borel subgroup B, we have the decomposition $S = \mathcal{O} \dot{\cup} Y$ where $\mathcal{O}$ is the B-orbit, and define the incidence variety

$$I_Y := \{g \in G^{\mathbb{C}} : g(\mathrm{c}\ell(\kappa)) \cap Y \neq \emptyset\}.$$

Observe that I_Y is a complex analytic subset of $G^{\mathbb{C}}$. Recall that for $g \notin I_Y$ the intersection $g(\mathrm{c}\ell(\kappa)) \cap S$ is finite.

Let U_S denote the subset of $G^{\mathbb{C}}$ defined by

$$U_S = \{g \in G^{\mathbb{C}} : g(p_0) \in S\}$$

and let Q be the $G^{\mathbb{C}}$-isotropy subgroup at $z_0 \in Z$. Then the evaluation map

$$U_S \to S, \text{ given by } g \mapsto g(p_0),$$

is a Q-principal bundle. Indeed, it suffice to show that Q acts transitively and freely on the fibers. So let $F_s = \{g \in U_S : g(p_0) = s\}$ denote a typical fiber over $s \in S$. Fix $g \in F_s$ then we have the following identification of Q and F_s; $Q \ni h \mapsto g.h \in F_s$. Consequently, the right Q-action on the fibers is transitive and free.

Since S is irreducible, it follows that U_S is an irreducible complex analytic subset of $G^{\mathbb{C}}$.

Set $\mathcal{E} := U_S \cap I_Y$ and observe that $\mathcal{E}$ is a proper analytic subset of U_S. Since U_S is irreducible, it follows that $\mathcal{E}$ is nowhere dense.

Denote by D_s the subset of S such that the fiber over $s \in S$, is contained in $\mathcal{E}$, that is,

$$D_s := \{s \in S : F_s \subset \mathcal{E}\}.$$

This defines a closed proper complex analytic subset of S which is nowhere dense as shown in the following

Lemma 6.1. *The set D_S is a proper complex analytic subset of S.*

Proof. Observe that the map $\pi : U_S \to S$ is a bundle with connected fibers and that the set $\mathcal{E}$ is a closed analytic subset in U_S. Let $\mathcal{E} = \cup \mathcal{E}_i$ be the decomposition of $\mathcal{E}$ in to irreducible components. For each i, let A_i be the subset of $\mathcal{E}$ such that the fiber of $\pi|_{\mathcal{E}_i}$ at $g \in \mathcal{E}$ is the same as the π-fiber at that point. If A_i is nonempty, then it defines a closed analytic subset of $\mathcal{E}$ since $rank_g(\pi|\mathcal{E}_i) := dim_g \mathcal{E}_i - dim_g \pi^{-1}(\pi(g))$ is minimal. Thus $A := \cup A_i$ is

a closed analytic subset in $\mathcal{E}$. Now $\pi(A)$ is closed and the restriction of π to each irreducible component of A has constant rank, therefore it follows that $\pi(A)$ is a finite union of analytic sets and thus analytic. $\square$

Proposition 6.2. *Given a point $g \in U_S$ and a sequence $\{p_n\} \subset S \setminus D_S$ converging to $p = g(p_0) \in Y$, there exists a sequence of transformations $\{g_n\} \subset U_S \setminus \mathcal{E}$ with g_n converging to g and $g_n(p_0) = p_n$.*

Proof. Let $\{U_n\}$ be a sequence of open subsets of U_S contracting to g, that is, $U_n \subset U_{n+1}$ for all n and $\bigcap U_n = g$. Since $\pi : U_S \to S$ is an open mapping, it follows that $V_n := \pi(U_n)$ is a sequence of open neigborhoods of p. Consequently, we can renumber the sequence p_n such that $p_n \in V_n$ for each n. Since the set $\mathcal{E}$ is a nowhere dense analytic subset of U_S (see, Lemma 6.1) and $p_n \notin D_S$, it follows that $\mathcal{E} \cap (F_{p_n} \cap U_n)$ is nowhere dense in $F_{p_n} \cap U_n$. Here, F_{p_n} denotes for each n, the fiber over p_n. We can therefore choose $g_n \in (F_{p_n} \cap U_n) \setminus \mathcal{E}$ such that $g_n(p_0) = p_n$. This yields a sequence $\{g_n\} \in U_S \setminus \mathcal{E}$ converging to g as required. $\square$

7 Hypersurfaces complementary to $C(\gamma)$

Our aim here is to show the existence of a complex B-invariant hypersurface in the complement of $C(\gamma)$ by constructing a function $f \in \Gamma(S, \mathcal{O}(*Y))$ and showing that the polar set of the trace transform $\mathcal{P}(Tr(f))$, lies outside of the cycle space. We continue on in the setup and with the notation of the previous chapter.

Proposition 7.1. *If $g \in I_Y$ and there is a sequence $\{g_n\} \subset G^{\mathbb{C}} \setminus I_Y$ with $g_n \to g$ and $p_n \subset g_n(C_0) \cap \mathcal{O}$ such that $p_n \to p \in Y$, then there exists $f \in \Gamma(S, \mathcal{O}(*Y))$ with $p \in \mathcal{P}(f)$ and $g \in \mathcal{P}(Tr(f))$.*

Proof. Through the map

$$\pi : C(\gamma) \to C(\gamma)/G^{\mathbb{C}}_{C_0}, \text{ given by } g \mapsto \pi(g) = C := g(\mathrm{c}\ell(\kappa)),$$

we identify the sequence $\{g_n\} \in G^{\mathbb{C}} \setminus I_Y$ converging to $g \in I_Y$ with a sequence of cycles $C_n := g_n(C_0) \notin I_Y/G^{\mathbb{C}}_{C_0}$ converging to $g(C_0) \in I_Y/G^{\mathbb{C}}_{C_0}$ with $\{p_n\} \subset g_n(C_0) \cap S$ converging to $p \in Y$.

Now embed S (see Prop. 5.3) into some projective space so that Y is the intersection of S with the hyperplane at infinity. Since $g_n(C_0) \cap S \subset \mathcal{O}$ and

this intersection is finite, we have that it is equal to a sequence of the form $\{p_1^n, \dots p_{k_n}^n\} \subset \mathcal{O}$ with at least some $p_i^n \to p \in Y$, since $p_n \to p \in Y$. Without loss of generality, assume $p_n = p_1^n \to \infty$. This is equivalent to $p_1^n \to \infty$ now considered as a sequence in $\mathbb{C}^m$, the complement of the hyperplane $\{z_0 = 0\}$ (see Cor.5.4).

Since the sequences $\{p_i^n\}$ above may be replaced by subsequences, we may assume that $k_n = k$ is a constant. Let $(z_1, \dots, z_m)$ denote standard coordinates in $\mathbb{C}^m$. Then with respect to this coordinate system, $p_i^n = (z_{i1}^n, \dots z_{im}^n)$ and since $p_1^n \to \infty$, we may assume again without loss of generality that $z_{11}^n \to \infty$ as a sequence of complex numbers.

Let $a^n := (a_1^n, \dots, a_k^n) \in \mathbb{C}^k$ be the sequence of first coordinates of p_i^n, in other words, $a_1^n = z_{11}^n, \dots, z_{k1}^n$. With this translation, the goal is to find a polynomial $P = P(z)$ of one variable so that its restriction to $\mathcal{O}$ when regarded as a function $f(z) := P(z_1)$ on $\mathbb{C}^m$ satisfies

$$\lim_{n\to\infty} \sum_i P(a_i^n) = \infty.$$

Let $\mathfrak{S}_k$ be the symmetric group acting on $\mathbb{C}^k$ as usual. Since the natural projection $\pi : \mathbb{C}^k \to Sym_k(\mathbb{C}) := \mathbb{C}^k/\mathfrak{S}_k$ is proper, it follows that $b^n := \pi(a^n) = (a_1^n, \dots a_k^n)$ (still denoted by the same coordinates), diverges in $Sym_k(\mathbb{C})$. Now since the algebraic variety $Sym_k(\mathbb{C})$ is affine, it follows that there is a regular function $R \in \mathcal{O}_{alg}(Sym_k(\mathbb{C}))$ with $R(b^n) \to \infty$. But the regular functions on $Sym_k(\mathbb{C})$ are generated by the Newton polynomials

$$N_\alpha := \sum_i w_i^\alpha,\ \alpha = 1, \dots, k,$$

and consequently, R is a polynomial $R = Q(N_0, \dots, N_k)$ in the Newton polynomials. Since $R(b^n) \to \infty$, it follows that at least one of the Newton polynomials, N_p, must be unbounded along the sequence $\{b^n\}$.

Consequently, after going to a subsequence,

$$\lim_{n\to\infty} N_p(b^n) = \lim_{n\to\infty} \sum_i (z_i^n)^p = \infty.$$

Therefore the function $f(z) = P(z_1, \dots, z_m) = z_1^p$ on $\mathbb{C}^m$ is such that $\lim_{n\to\infty} (\mathcal{T}_\psi(f)) = \infty$ and so f is the required function. $\square$

The following is a converse of Prop. 7.1.

Proposition 7.2. *If $g \in \mathcal{P}(\mathcal{T}(f))$ and the sequence $\{g_n\} \subset G^{\mathbb{C}} \setminus I_Y$ converges to g, then there exists a sequence $\{p_n\} \subset g_n(C_0) \cap S$ such that $p_n \to p \in Y$.*

Proof. We recall that $\mathcal{T}(f)$ is defined by averaging f over the intersection $g(C_0) \cap S$. Therefore, If $g_n \to g \in \mathcal{P}(\mathcal{T}(f))$, then it follows that $\mathcal{T}((f)(g_n(C_0)))$ also converges to $\mathcal{T}(f)g(C_0)$ which is infinite since $g \in \mathcal{P}(\mathcal{T}(f))$. If all the elements of the sets $g_n(C_0) \cap S$ were bounded away from Y, then $\mathcal{T}(f)$ would be bounded. Hence there is a point $p \in Y$ and a sequence $\{p_n\} \in g_n(C_0) \cap S$ such that $p_n \to p$. □

Now for (γ, κ) a dual pair and $p_0 \in \gamma \cap \kappa$ a base point, define the following subset of U_S;

$$U_Y := \{g \in G^{\mathbb{C}} : g(p_0) \in Y\}.$$

Proposition 7.3. *Let $g \in U_Y$, then there exists $f \in \Gamma(S, \mathcal{O}(*Y))$ such that $g \in \mathcal{P}(\mathcal{T}(f))$ and $g \notin C(\gamma)$.*

Proof. Since $g(p_0) = p \in Y$ and D_S is nowhere dense in S, there is a sequence of points $\{p_n\} \subset S \setminus D_S$ with p_n converging to p. Now by Prop. 6.2, there exists a sequence of transformations $\{g_n\} \subset U_S \setminus \mathcal{E}$ with g_n converging to g such that $g_n(p_0) = p_n$. Thus the first statement follows from Prop. 7.1.

Since $g \in \mathcal{P}(\mathcal{T}_\psi(f))$, it follows that $g_n(\mathrm{c}\ell(\kappa)) \cap S$ is a finite set say, $\{p_1^n, \ldots, p_k^n\}$ and is contained in $\mathcal{O} \cap \gamma$. Since g_n converges to $g \in \mathcal{P}(\mathcal{T}_\psi(f))$, at least one of the sequences $\{p_i^n\}$ converges to a point in Y. Assume that $p_1^n \to p \in Y$. It now follows from Cor. 3.7 that g is not contained in $C(\gamma)$. □

Observe that since I_Y is $K^{\mathbb{C}}$-invariant, it follows that if $g(p_o) \in I_Y$, then there exists $k \in K^{\mathbb{C}}$ such that $g(k(p_0)) \in U_Y$. Consequently, $g \in U_Y.K^{\mathbb{C}}$.

Corollary 7.4. *If $g \in U_Y.K^{\mathbb{C}}$, then I_Y is locally 1-codimensional at g and is contained in the complement of $C(\gamma)$.*

Lemma 7.5. *The set $U_Y.K^{\mathbb{C}}$ is dense in I_Y.*

Proof. For any given $g \in I_Y$, there exists an arbitrarily small $h \in G^{\mathbb{C}}$ so that $hg \in U_Y.K^{\mathbb{C}}$. In particular $U_Y.K^{\mathbb{C}} \cap I_Y$ is dense in I_Y. □

Corollary 7.6. *The set I_Y is 1-codimensional and contained in the complement of $C(\gamma)$.*

Proof. We have shown above that if $g \in U_Y$, then $g \notin C(\gamma)$. Therefore, $g \notin C(\gamma)$ if $g \in U_Y.K^{\mathbb{C}}$. Furthermore, for any $g \in U_Y.K^{\mathbb{C}}$ there exists $f \in \Gamma(S, \mathcal{O}(*Y))$ such that $g \in \mathcal{P}(\mathcal{T}(f))$. Consequently, $U_Y.K^{\mathbb{C}} \cap I_Y$ is contained in the complement of $C(\gamma)$ and is 1-codimensional at each point of I_Y. Since $U_Y.K^{\mathbb{C}} \cap I_Y$ is dense in I_Y, it follows that I_Y is 1-dimensional and is contained in the complement of $C(\gamma)$. □

8 Cycle spaces of nonclosed orbits

Let $B \subset G^{\mathbb{C}}$ denote as usual an Iwasawa-Borel subgroup. Then B has only finitely many orbits in $\Omega := G^{\mathbb{C}}/K^{\mathbb{C}}$. If $x_0 := 1K^{\mathbb{C}}$ denote the base point in Ω, then the orbit $B.x_0$ is open in Ω, since the complexification of the Iwasawa decomposition $G = K.A.N$ of G is open in $G^{\mathbb{C}}$. The complement of the open B-orbit in Ω therefore consists of a union of a finite number of B-invariant complex hypersurfaces. Let H be such a hypersurface invariant for some fixed B, then the family $\{gH\}_{g\in G}$ consists of G-translates of H. For some Iwasawa decomposition of G, the hypersurface H is AN-invariant since $AN \subset B$ and consequently, this family is equivalent to the family $\{kH\}_{k\in K}$. We denote by Ω_H the G-invariant domain defined by H in the following way;

$$\Omega_H := (\Omega \setminus \bigcup_{k\in K} (kH))^0,$$

the connected component containing the base point x_0 in Ω.

We will first handle a certain situation which is present in all nonhermitian cases and many Hermitian cases. For this we recall the notation in the Hermitian setting.

Associated to the compact symmetric spaces X_+ and X_- are bounded symmetric domains realized as G-orbits of the neutral points in the following way. Let x_+ be the neutral point in X_+ and x_- the neutral point in X_- i.e., $x_+ = e.P_+ \in X_+$ and $x_- = eP_- \in X_-$. Let us denote by $\mathcal{B}$ the bounded symmetric space G/K with the complex structure of $G.x_-$ and $\bar{\mathcal{B}}$ the bounded symmetric space G/K with the complex structure of $G.x_+$.

As usual, let $(\gamma, \kappa) \in Orb_Z(G) \times Orb_Z(G^{\mathbb{C}})$ be a dual pair considered here in the Hermitian case for an arbitrary flag manifold Z. Recall that by the $K^{\mathbb{C}}$-invariance of the base cycle $C_0 = \mathrm{c}\ell(\kappa)$, the cycle space $C(\gamma)$ is right $K^{\mathbb{C}}$-invariant. If C_0 is only $K^{\mathbb{C}}$-invariant, we regard the cycle space as $C(\gamma)/K^{\mathbb{C}} \subset$

$G^{\mathbb{C}}/K^{\mathbb{C}}$. If C_0 is either P_+- or P_-- invariant, then we regard the cycle space as $C(\gamma)/P_+$ or $C(\gamma)P_-$.

We also recall that the Iwasawa-Borel subgroup B acts on the symmetric spaces $X_+ = G^{\mathbb{C}}/P_+$ and $X_- = G^{\mathbb{C}}/P_-$. Since $b_2(X) = 1$, there is a unique B-invariant hypersurface H_0 in the complement of $\mathcal{B}$. We will maintain notation of the previous sections and recall that we have the B-invariant hypersurface I_Y containing $\mathcal{P}(\mathcal{T}_\psi(f))$, the polar set of the trace transform $\mathcal{T}_\psi(f)$, constructed in Section 7. Moreover, I_Y is in the complement of $C(\gamma)$.

Now define H to be the maximal B-invariant hypersurface in the complement of the cycle space $C(\gamma)$. In the proof of the main results, it will be important if the hypersurface H is a π_+- (or π_-)-lift of H_0 from X_+ (or X_-) or not and so these two cases will be distinguished.

Since the $G^{\mathbb{C}}$-isotropy subgroup $G^{\mathbb{C}}_{C_0} = \{g \in G^{\mathbb{C}} : g(C_0) = C_0\}$ of the base cycle C_0 contains $K^{\mathbb{C}}$, the orbit $G^{\mathbb{C}}.C_0$ is either $G^{\mathbb{C}}/\tilde{K}^{\mathbb{C}}$, where $\tilde{K}^{\mathbb{C}}$ is a finite extension of $K^{\mathbb{C}}$ or it is one of the compact Hermitian symmetric spaces $G^{\mathbb{C}}/P_+$ or $G^{\mathbb{C}}/P_+$.

8.1 Case I: H is not a lift

Here we consider the case when there exists a maximal B-invariant hypersurface H in the complement of $C(\gamma)$ which is not a lift. This means that for every Iwasawa-Borel group B the maximal B-invariant hypersurface is neither of the form $H = \pi_+^{-1}(H_+)$ nor of the from $H = \pi_-^{-1}(H_-)$, where $\pi_\pm : G^{\mathbb{C}}/K^{\mathbb{C}} \to G^{\mathbb{C}}/P_\pm = X_\pm$ are the standard projections. Here H_+ (resp. H_-) is the unique B-invariant hypersurface in X_+ (resp. X_-).

Of course in the nonhermitian case there are no such projections and therefore this imposes no condition.

Since H is not a lift and Ω_H contains the cycle space, the orbit $G^{\mathbb{C}}.C_0$ in $\mathcal{C}_q(Z)$ is $G^{\mathbb{C}}/\tilde{K}^{\mathbb{C}}$. Making use of the finite covering map $\pi : G^{\mathbb{C}}/K^{\mathbb{C}} \to G^{\mathbb{C}}/\tilde{K}^{\mathbb{C}}$, we lift the cycle space to $C(\gamma)/K^{\mathbb{C}}$ which is an open subset of $G^{\mathbb{C}}/K^{\mathbb{C}}$ and still denote it by $C(\gamma)$.

This will allow us to be able to compare $C(\gamma)$ with the domain Ω_{AG} which is contained in $G^{\mathbb{C}}/K^{\mathbb{C}}$. We state the following result in this context.

Theorem 8.1. *If the maximal B-invariant hypersurface H in the complement of $C(\gamma)$ is not a lift, then*

$$\Omega_{AG} = C(\gamma) = \Omega_H.$$

In order to prove this theorem, we first state some known results concerning an open G-orbit $\gamma_{open} \in Orb_Z(G)$.
Firstly, it has been shown in [GM] that the intersection of all cycle spaces $C(\gamma)$ as γ ranges over $Orb_Z(G)$ and Q ranges over all parabolic subgroups of $G^{\mathbb{C}}$ is the same as the intersection of all cycle spaces $C(\gamma_{open})$ for all open G-orbits in $Z = G^{\mathbb{C}}/B$, for B a Borel subgroup of $G^{\mathbb{C}}$. That is

Lemma 8.2. *([GM]).*

$$\bigcap_{\substack{\gamma \in Orb_Z(G) \\ Z = G^{\mathbb{C}}/Q}} C(\gamma) = \bigcap_{\substack{\gamma\, open \\ Z = G^{\mathbb{C}}/B}} C(\gamma).$$

If G is of Hermitian type the following result is also known

Theorem 8.3. *([HW1],[WZ]). If G is of Hermitian type and $\gamma \in Orb_Z(G)$ is open, then either*

1. *the base cycle κ is P_+-invariant (resp. P_--invariant) and $C(\gamma) = \bar{\mathcal{B}}$ (resp. $\mathcal{B}$)*
   or

2. *the base cycle is only invariant by $K^{\mathbb{C}}$ and $C(\gamma) = \mathcal{B} \times \bar{\mathcal{B}}$.*

Since $\Omega_{AG} = \mathcal{B} \times \bar{\mathcal{B}}$ in the Hermitian case ([BHH]), we have the following consequence.

Corollary 8.4. *If G is of Hermitian type and $\gamma \in Orb_Z(G)$ is open, then*

$$\bigcap_{\substack{\gamma\, open \\ Z = G^{\mathbb{C}}/B}} C(\gamma) = \mathcal{B} \times \bar{\mathcal{B}} = \Omega_{AG}.$$

Proof. Although there will always be open orbits with cycle spaces of the second type above, it suffices to note that

$$\pi_-^{-1}(\mathcal{B}) \cap \pi_+^{-1} \bar{\mathcal{B}} = \mathcal{B} \times \bar{\mathcal{B}}$$

which is embedded as Ω_{AG} in $G^{\mathbb{C}}/K^{\mathbb{C}}$. $\square$

If G is not of Hermitian type, then we have the following result ([FH]) for open G-orbits in $Z = G^{\mathbb{C}}/B$.

Theorem 8.5. *([FH]). If G is not Hermitian, then*

$$C(\gamma) = \Omega_{AG}$$

for every open G-orbit $\gamma \in Orb_Z(G)$ and Z any flag manifold.

Corollary 8.6. *In both the Hermitian and non-Hermitian cases,*

$$\bigcap_{\substack{\gamma\, open \\ Z=G^{\mathbb{C}}/B}} C(\gamma) = \Omega_{AG}.$$

Corollary 8.7. *For all $\gamma \in Orb_Z(G)$,*

$$\Omega_{AG} \subset C(\gamma).$$

Proof. Apply Lemma 8.2. □

This immediately implies that if H is a maximal B-invariant hypersurface which lies outside $C(\gamma)$, then

$$\Omega_{AG} \subset C(\gamma) \subset \Omega_H$$

.

Proof of Theorem 8.1. This is now an immediate consequence of the work in [FH]. There as a first step it is shown that if H is not a lift from either $G^{\mathbb{C}}/P_+$ or $G^{\mathbb{C}}/P_-$, then the domain Ω_H is Kobayashi hyperbolic. The following main result of [FH] then completes our proof: If $\hat{\Omega}$ is a G-invariant, Kobayashi hyperbolic Stein domain in Ω which contains Ω_{AG}, then $\hat{\Omega} = \Omega_{AG}$. □

Theorem 8.1 proves our main result Theorm 1.1 for non-closed orbits in the case where tha base cycle $C_0 = \mathrm{c}\ell(\kappa)$ is neither P_-- nor P_--invariant .

The following is a consequence of Theorem 8.1.

Corollary 8.8. *If G is not of Hermitian type, then*

$$C(\gamma) = \Omega_{AG}$$

for all $\gamma \in Orb_Z(G)$.

Proof. Since G is non-Hermitian, no H is a lift. □

8.2 Case II: Every H is a lift

We now consider the case where the maximal B-invariant hypersurface $H = \pi^{-1}(H_+)$ is a lift from $G^{\mathbb{C}}/P_+$, where π_+ is the natural projection. Of course the discussion is the same if every H is a lift from $G^{\mathbb{C}}/P_-$.

We begin by proving the following

Theorem 8.9. *If $c\ell(\kappa)$ is not P_+-invariant, then no Schubert variety $S = \mathcal{O} \dot{\cup} Y \in \mathcal{S}_\kappa$ defines a maximal B-invariant hypersurface H which is a lift from $G^{\mathbb{C}}/P_+$.*

Proof. Suppose to the contrary that there is some Schubert variety $S = \mathcal{O} \dot{\cup} Y \in \mathcal{S}_\kappa$ defining $H = \pi^{-1}(H_+)$ which is lift. This is equivalent to the domain $\Omega_H = \pi_+^{-1}(\Omega_{H_+})$ being a lift.

Let $x_0 \in \gamma$ be the base point with $\kappa = K^{\mathbb{C}}.x_0$. Since κ is not P_+-invariant, $c\ell(P_+.x_0)$ contains $c\ell(\kappa)$ as a proper subvariety. Now the intersection $c\ell(\kappa) \cap S \subset \mathcal{O}$ and is transversal in Z. Thus every component of $P_+.x_0 \cap \mathcal{O}$ is positive dimensional. Since $\mathcal{O} = \mathbb{C}^{m(\mathcal{O})}$ is affine, every such component has at least one point of Y in its closure.

Thus for every arbitrarily small neigborhood U of the identity in $G^{\mathbb{C}}$ there exists $h \in P_+$, and $g \in U$ with $gh.x_0 \in Y$. Consequently, $gh \in U_S$ and it follows from Prop. 7.3 that $ghK^{\mathbb{C}}$ is in the maximal B-invariant hypersurface H. Hence $ghK^{\mathbb{C}} \notin \Omega_H = \pi_+^{-1}(\Omega_{H_+})$.

On the other hand $C(\gamma) \subset \Omega_H$ and $C(\gamma)$ contains an open neighborhood U of the identity. Consequently, $ghK^{\mathbb{C}} \subset \Omega_H$ for every $g \in U$ and $h \in P_+$. Thus we have reached a contradiction, and therefore no Schubert variety $S \in \mathcal{S}_\kappa$ defines $H = \pi_+^{-1}(H_+)$. $\square$

It therefore follows that if $c\ell(\kappa)$ is neither P_+- nor P_--invariant, the domain Ω_H is Kobayashi hyperbolic [FH]. Thus the proof of our main result Theorem 1.1 for non-closed orbits in this case is completed just like in the proof of Theorem 8.1.

By taking contrapositions in the above theorem, we obtain

Corollary 8.10. *If the maximal B-invariant hypersurface $H = \pi_+^{-1}(H_+)$ is a lift, then $c\ell(\kappa)$ is P_+-invariant.*

Corollary 8.11. *If the maximal B-invariant hypersurface $H = \pi_+^{-1}(H_+)$ (resp. $H = \pi_-^{-1}(H_-)$) is a lift from $G^{\mathbb{C}}/P_+$ (resp. $G^{\mathbb{C}}/P_-$), then $C(\gamma) = \bar{\mathcal{B}}$ (resp. $C(\gamma) = \mathcal{B}$).*

Proof. We have seen above that under this assumption $c\ell(\kappa)$ is P_+-invariant. This implies that $g(c\ell(\kappa))$ is gP_+g^{-1}-invariant. Consequently, if $g \in C(\gamma)$, then $gP_+g^{-1}.g \subset C(\gamma)$, that is, $C(\gamma)$ is right P-invariant. Thus $C(\gamma)$ may be regarded as a domain in $G^{\mathbb{C}}/P_+$. Since $\bar{\mathcal{B}} \subset G^{\mathbb{C}}/P_+$ is a G-orbit, it follows that $\bar{\mathcal{B}} \subset C(\gamma)$. Since $C(\gamma) \subset \Omega_{H_+}$ and we know that $\Omega_{H_+} = \bar{\mathcal{B}}$ (see for example [H]), the result follows. $\square$

This completes the proof of our main result Theorem 1.1 for non-closed orbit.

As a consequence of the work in this and the previous subsections we now have the following result.

Theorem 8.12. *Suppose γ is a nonclosed G-orbit. If G is of Hermitian type and $c\ell(\kappa)$ is neither P_+- nor P_--invariant, then*

$$C(\gamma) = \Omega_{AG}. \tag{1}$$

If γ is nonclosed and $c\ell(\kappa)$ is P_+-invariant (resp. P_--invariant), then $C(\gamma) = \bar{\mathcal{B}}$ (resp. $C(\gamma) = \mathcal{B}$).

Note that if $c\ell(\kappa)$ is P_+- or P_--invariant, then the orbit $G^{\mathbb{C}}.C_0$ in the cycle space $\mathcal{C}_q(Z)$ is $G^{\mathbb{C}}/P_+$ or $G^{\mathbb{C}}/P_-$. Thus the latter statement, $C(\gamma) = \bar{\mathcal{B}}$ or $C(\gamma) = \mathcal{B}$, is a statement in $\mathcal{C}_q(Z)$.

The former statement must be interpreted. In that case $G^{\mathbb{C}}.C_0 = G^{\mathbb{C}}/\tilde{K}^{\mathbb{C}}$, where $\tilde{K}^{\mathbb{C}}$ is possibly a finite extension of $K^{\mathbb{C}}$. If we regard the sets Ω_H, which are defined by incidence geometry, as being in $\mathcal{C}_q(Z)$, then the cycle space statement is $C(\gamma) = \Omega_H$ in $\mathcal{C}_q(Z)$.
However, by the main result of ([FH]) the lift of Ω_H in $G^{\mathbb{C}}/K^{\mathbb{C}}$ is Ω_{AG}, and, since Ω_{AG} is a cell, this lift is biholomorphic. Thus in this sense we write $C(\gamma) = \Omega_{AG}$ in $G^{\mathbb{C}}/K^{\mathbb{C}}$.

9 Cycle spaces of closed orbits

Here we consider the case of the closed G-orbit $\gamma_{c\ell}$ and its dual κ_{op} which is the open $K^{\mathbb{C}}$-orbit in Z. Duality is just the statement that $\kappa_{op} \supset \gamma_{c\ell}$.

We begin by recalling the behavior of duality with respect to the partial ordering of orbits defined by the closure operation. For this, if $\mathcal{O}_1$ and $\mathcal{O}_2$ are orbits with $\mathcal{O}_2 \subset \mathrm{c}\ell(\mathcal{O}_1) \setminus \mathcal{O}_1$, we write $\mathcal{O}_1 < \mathcal{O}_2$. The following is a well-known aspect of the duality theory.

Proposition 9.1. *If* (γ_1, κ_1) *and* (γ_2, κ_2) *are dual pairs, then*

$$\gamma_1 < \gamma_2 \;\Leftrightarrow\; \kappa_2 < \kappa_1 .$$

Sketch of proof. Suppose that $\gamma_1 < \gamma_2$. As a consequence, $\gamma_1 \cap \kappa_2 \neq \emptyset$. Recall that $\kappa_1 \cap \gamma_1$ is realized as a strong deformation retract by the gradient flow φ_t of the norm of a certain moment map (see, e.g., [BL]). This flow is K-invariant and tangent to all $K^{\mathbb{C}}$- and G-orbits.

Now take $p \in \kappa_2 \cap \gamma_1$ and let $q := \lim\limits_{t\to\infty} \varphi_t(p) \in \kappa_1 \cap \gamma_1$. Since $\mathrm{c}\ell(\kappa_2)$ is invariant under the flow, it follows that $\kappa_1 = K^{\mathbb{C}}.q \subset \mathrm{c}\ell(\kappa_2)$. By assumption $\gamma_1 \neq \gamma_2$. Thus $\kappa_1 \subset \mathrm{c}\ell(\kappa_2) \setminus \kappa_2$ as required.
The converse implication goes in exactly the same way. □

This result will now be used in the following special situation. Let

$$bd(\kappa_{op}) = Z \setminus \kappa_{op} = A_1 \cup \ldots \cup A_k$$

be the decomposition of the boundary $bd(\kappa_{op})$ of κ_{op} as a union of its irreducible components. In each A_j there is a unique, Zariski dense open $K^{\mathbb{C}}$-orbit κ_j with dual G-orbit γ_j, $j = 1, \ldots, k$.

Corollary 9.2. *For every* j *it follows that* $c\ell(\gamma_j) = \gamma_j \,\dot{\cup}\, \gamma_{c\ell}$.

Proof. If γ is a G-orbit contained in $\mathrm{c}\ell(\gamma_j) \setminus \gamma_j$, then its dual $K^{\mathbb{C}}$-orbit κ satisfies $\kappa > \kappa_j$. But $\kappa = \kappa_{\mathrm{op}}$ is the only $K^{\mathbb{C}}$-orbit with this property. □

Corollary 9.3. *For* γ_j *as above,*

$$C(\gamma_{c\ell}) \subset C(\gamma_j).$$

Proof. If $g \in \mathrm{bd}(C(\gamma_j))$, then $g(\mathrm{c}\ell(\kappa_j)) \cap \mathrm{bd}(\gamma_j) \neq \emptyset$. Thus $g(\mathrm{c}\ell(\kappa_j)) \cap \gamma_{\mathrm{c}\ell} \neq \emptyset$. In particular, $g \notin C(\gamma_{\mathrm{c}\ell})$. □

Theorem 9.4. *For every* $Z = G/Q$ *it follows that*

$$C(\gamma_{c\ell}) = \Omega_{AG}.$$

Proof. By the above Corollary and Cor. 8.7, for all j

$$\Omega_{AG} \subset C(\gamma_{c\ell}) \subset C(\gamma_j).$$

If $C(\gamma_j) = \Omega_{AG}$ for some j, then the proof is finished.

In the Hermitian case, if, e.g., $C(\gamma_1) = \pi_-^{-1}(\mathcal{B})$ and $C(\gamma_2) = \pi_+^{-1}(\bar{\mathcal{B}})$, then $C(\gamma_1) \cap C(\gamma_2) = \Omega_{AG}$ and the proof is finished in that case as well.

Thus we may assume that $C(\gamma_j) = \pi_-^{-1}(\mathcal{B})$ for all j, or equivalently that $c\ell(\kappa_j)$ is P_--invariant for all j. But this is in turn equivalent to κ_{op} being P_--invariant.

However, κ_{op} can not be P_--invariant. To see this, note that if it were invariant, then $C(\gamma_{c\ell})$ would be P_--invariant. Since $\Omega_{AG} \subset C(\gamma_{c\ell})$, this would imply that $P_-.\Omega_{AG} \subset C(\gamma_{c\ell})$.

But the P_--orbit of a generic point in Ω_{AG} is Zariski open in Ω. Consequently, if κ_{op} were P_--invariant, it would follow that $C(\gamma_{c\ell})$ would contain a Zariski open subset of Ω. By the identity principle, this is contrary to the complement of $C(\gamma_{c\ell})$ being nonempty and G-invariant. $\square$

This completes the proof of our main theorem, Theorem 1.1.

10 References

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